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THEORETICAL FLEXURAL RESPONSE OF A PRESSURIZED CYLINDRICAL MEMBRANE

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Abstract—An analysis of the finite bending of an inflated circular cylindrical membrane is presented. Wrinkling of the membrane is taken into account by using a relaxed energy derived from a strain energy function of Varga type. First integrals of the equilibrium equations are obtained and used to reduce the analysis to quadratures. The solutions are then used to study various features of the deformation and to generate the equilibrium moment–curvature relations for three values of the inflation pressure.

1. INTRODUCTION

The problem of flexure of a pressurized cylindrical membrane was formulated and solved by Stein and Hedgepeth (1961) in the context of a theory for infinitesimal deformations. Their model accounts for wrinkling of the membrane by incorporating a suitably modified version of Reissner's *tension-field theory* (Reissner, 1938). This is an approximate equilibrium theory for the stress and deformation associated with fine scale wrinkling of a thin plate with vanishingly small bending stiffness. In this theory it is assumed that the wrinkles are continuously distributed over a smooth surface and coincide with the trajectories of the active principal stress. The second principal stress is assumed to vanish identically. A finite-deformation version of this theory was formulated by Wu and Canfield (1981) for application to plane-stress problems.

For isotropic elastic membranes with no bending stiffness, Pipkin (1986) showed that all of the basic hypotheses of tension-field theory follow as consequences of the principle of minimum potential energy. In particular, states of strain associated with unstable compressive stresses in conventional membrane theory may instead be constructed as limits of energy-minimizing sequences of deformations involving closely spaced wrinkles. Because of the absence of bending stiffness, there is no energetic penalty associated with spacing the wrinkles more and more closely together. The wrinkles are continuously distributed in the limit and the resulting configuration is, in general, perfectly smooth and free of compressive stress. Pipkin (1986) used these constructions to derive a *relaxed* strain energy that automatically accounts for states with continuously distributed wrinkles. Subsequently, Steigmann (1990a) used this energy to develop a general tension-field theory for finite deformations of arbitrarily curved membranes composed of isotropic materials.

To predict the details of the distribution, spacing and amplitude of the wrinkles, one must resort to a theory that accounts for the strain energy due to bending (see e.g. Hilgers and Pipkin, 1992). However, if detailed information of this kind is not required, then a pure membrane theory based on a relaxed energy density furnishes a much simpler alternative. The latter theory is used in the present paper to extend the Stein-Hedgepeth analysis to finite deformations. Specifically, we solve the problem of a finite flexural deformation superposed on a radial expansion and axial extension of a sealed and pressurized cylinder. We assume the deformation to be the same at each cross-section, and thus independent of the axial coordinate along the length of the cylinder. Thus we obtain a pair of ordinary differential equations for the coordinates of points on the contour of the deformed cross-section. The independent variable is the azimuthal angle on the undeformed cylinder. Various versions of this and related problems are well known in the context of shell theory (see Reissner and Weinitschke, 1963; Emmerling, 1984; Axelrad, 1985). The book by Libai and Simmonds (1988) contains a comprehensive discussion of these problems together with an extensive bibliography.

We begin in Section 2 with a brief account of kinematics and stress in isotropic elastic membranes. Locally stable equilibria are defined to be minimizers of a certain potential energy. In particular, we consider the problem of inflation by a fixed pressure and assume that the curvature of the tube is prescribed in a sense that we define subsequently (Section 6). The appropriate form of the potential energy is given in Section 2.

In Section 3 we discuss the construction of the relaxed energy and the form that it takes in wrinkled parts of the membrane. Our analysis is based on the Varga strain energy function (Varga, 1966). This material model is known to provide fairly good quantitative agreement with data on rubber at the moderate strain levels present in our solution. Moreover, its use greatly simplifies much of the algebraic detail.

A kinematical analysis of flexure is presented in Section 4. Following a discussion of the radial expansion and axial extension of a pressurized cylinder in Section 5, we obtain the Euler–Lagrange equations that describe superposed flexural deformations in Section 6. These are derived from an expression for the potential energy per unit initial length of the cylinder. The contribution from the pressure loading is based on the volume occupied by a unit length of the reference cylinder in the deformed configuration. Thus the pressure acting on the closed ends of the membrane is replaced by the pressure acting on the projected cross-sectional areas.

We derive a pair of first integrals of the Euler–Lagrange equations and use them in Section 7 to reduce the analysis to quadratures. Kydoniefs (1969) used a similar approach to analyse axisymmetric deformations of initially cylindrical membranes. The values of the first integrals must be adjusted to satisfy certain auxiliary conditions. A numerical method for obtaining these values is discussed in Section 8. We find two equilibrium deformations for fixed values of the pressure and curvature. An energy comparison is used to support an argument that one of the solutions is unstable at any curvature.

Finally, in Section 9 we use the solution with smaller energy to compute the resultant cross-sectional force and moment as functions of curvature at fixed values of the pressure. The qualitative features of the moment–curvature response lead us to conjecture that our solution is unstable at curvatures exceeding some critical value depending on the pressure. We explain our rationale in Section 9. Indeed, our use of a relaxed energy notwithstanding, our methods cannot be used to prove stability at any curvature.

Regarding notation, we use subscripts to denote partial derivatives and primes to denote ordinary derivatives with respect to the variable indicated.

2. EQUILIBRIUM OF ISOTROPIC ELASTIC MEMBRANES UNDER PRESCRIBED PRESSURE

Our analysis of the flexure problem is based on the direct theory of elastic membranes, in which the membrane is considered to be a two-dimensional continuum endowed with a strain energy measured per unit area of a reference surface. We regard this framework as preferable to the conventional approach, based on descent from three-dimensional elasticity (see Green and Adkins, 1970), because it exposes more clearly the underlying variational structure of the theory. Detailed discussions of the direct theory can be found in Stoker (1964), Green *et al.* (1965), Naghdi (1972), Steigmann (1990a). The equivalence of the direct and conventional theories has been established by Naghdi and Tang (1977) for isotropic elastic materials that are either compressible or incompressible in bulk.

To describe a configuration of the membrane, we define a reference surface by the parametric representation $\mathbf{x}(\xi_1, \xi_2)$, $(\xi_1, \xi_2) \in S$, where S is a parameter plane. Then a deformation is a mapping of this surface onto a surface $\mathbf{y}(\xi_1, \xi_2)$ in three-dimensional space. The deformation gradient $\mathbf{F}(\xi_1, \xi_2)$ is the linear transformation defined by:

$$d\mathbf{y} = \mathbf{F}\, d\mathbf{x}. \tag{1}$$

It can be represented in the form (Pipkin, 1986)

$$\mathbf{F} = \lambda \mathbf{l} \otimes \mathbf{L} + \mu \mathbf{m} \otimes \mathbf{M}, \tag{2}$$

where λ and μ are the non-negative principal stretches, $(\mathbf{l}, \mathbf{m})$ are the orthonormal vectors of strain spanning the tangent plane of the surface $\mathbf{y}$ at the particle with coordinates (ξ_1, ξ_2), and $(\mathbf{L}, \mathbf{M})$ are the corresponding principal vectors on the tangent plane of the reference surface. The orientations of these surfaces are defined by the unit normals:

$$\mathbf{n} = \mathbf{l} \times \mathbf{m} \quad \text{and} \quad \mathbf{N} = \mathbf{L} \times \mathbf{M}. \tag{3a, b}$$

As a constitutive hypothesis, we take the membrane to be perfectly flexible, with a strain energy per unit area of the reference surface that depends in some way on the deformation gradient $\mathbf{F}$. If the membrane is isotropic and its response is insensitive to superposed rigid rotations, then the strain energy may be expressed as a symmetric function $W^*(\lambda, \mu)$ of the principal stretches (Naghdi and Tang, 1977). We further assume that the membrane is homogeneous in the sense that the energy measured per unit area of the reference surface does not depend explicitly on the parameters ξ_1 and ξ_2 (Ericksen, 1970).

The force transmitted across an arbitrary material arc $\mathrm{d}\mathbf{x}$, with unit normal $\boldsymbol{\nu}$ and length $\mathrm{d}s$ defined by $\boldsymbol{\nu}\,\mathrm{d}s = \mathrm{d}\mathbf{x} \times \mathbf{N}$, is $\mathbf{T}\boldsymbol{\nu}\,\mathrm{d}s$, where $\mathbf{T}$ is the Piola stress-resultant. For isotropic membranes, the stress-resultant has the representation (Steigmann, 1990a)

$$\mathbf{T} = W^*_\lambda \mathbf{l} \otimes \mathbf{L} + W^*_\mu \mathbf{m} \otimes \mathbf{M}, \tag{4}$$

where subscripts λ and μ are used to denote partial derivatives.

We define stable equilibria to be local minimizers of the potential energy

$$E[\mathbf{y}] = \iint_\Omega W^*(\lambda, \mu)\,\mathrm{d}A - L[\mathbf{y}], \tag{5}$$

where $\mathrm{d}A$ is the elemental area of the reference surface, $\Omega = \mathbf{x}(S)$ is the image of the (ξ_1, ξ_2) parameter plane on this surface, and $L[\cdot]$ is a load potential. For a closed, pressurized membrane with a net internal pressure of prescribed intensity P, the load potential is

$$L[\mathbf{y}] = PV[\mathbf{y}], \tag{6}$$

where $V[\cdot]$ is the volume enclosed by the deformed surface. If $\mathbf{n}(\xi_1, \xi_2)$ is the exterior unit normal to this surface, then:

$$V[\mathbf{y}] = \iint_\Omega \chi\,\mathrm{d}A; \quad \chi = \frac{1}{3} J\mathbf{y} \cdot \mathbf{n}, \tag{7a}$$

where

$$J = \lambda\mu \tag{7b}$$

is the deformed surface area per unit of initial surface area.

Typically, in problems with no prescribed kinematical data, invariance of the total energy under superposed rigid motions is imposed to obtain a restriction on the prescribed forces. For example, if edge tractions and distributed forces are prescribed, then invariance with respect to arbitrary translations requires that the net applied force vanish. For the case of dead loading, this condition places an *a priori* restriction on the data that must be satisfied if the energy is to be bounded below. However, in the present class of problems involving pressure loading alone, no restrictions arise from such arguments because the energy is automatically invariant. To see this we consider a superposed rigid deformation

$$\mathbf{y}(\xi_1, \xi_2) \to \mathbf{Q}\mathbf{y}(\xi_1, \xi_2) + \mathbf{c}, \tag{8}$$

where $\mathbf{Q}$ is a fixed rotation ($\mathbf{Q}^{-1} = \mathbf{Q}^T$, $\det \mathbf{Q} = 1$) and $\mathbf{c}$ is a fixed vector. The exterior unit normal to the surface transforms according to $\mathbf{n} \to \mathbf{Qn}$. Then

$$\mathbf{y} \cdot \mathbf{n} \to \mathbf{Qy} \cdot \mathbf{Qn} + \mathbf{c} \cdot \mathbf{Qn} = \mathbf{y} \cdot \mathbf{n} + \mathbf{Q}^T \mathbf{c} \cdot \mathbf{n}, \tag{9}$$

and (7a) gives

$$V[\mathbf{y}] \to V[\mathbf{y}] + \frac{1}{3} \mathbf{Q}^T \mathbf{c} \cdot \iint_{\Omega} J\mathbf{n} \, \mathrm{d}A. \tag{10}$$

The second term involves the integral of $\mathbf{n}$ over the surface $\mathbf{y}(\xi_1, \xi_2)$, and this vanishes because the surface is closed. Thus the load potential $L[\cdot]$ is invariant. Because the term involving strain energy in eqn (5) is already invariant, it follows that the total potential is invariant, identically, with respect to arbitrary rigid motions.

The equilibrium equations for a pressurized membrane are simply the Euler–Lagrange equations associated with the energy $E[\cdot]$. We obtain these directly in Section 6 by specializing (5–7) to a specific class of flexural deformations.

3. WRINKLING AND THE RELAXED ENERGY DENSITY: VARGA MATERIALS

If an equilibrium deformation $\mathbf{y}(\xi_1, \xi_2)$ is continuous and continuously differentiable in S and minimizes the energy $E[\cdot]$, then the strain energy, expressed as a function of the deformation gradient, is locally convex with respect to rank-one perturbations $\mathbf{u} \otimes \mathbf{v}$ at the gradient $\mathbf{F}(\xi_1, \xi_2)$ of the minimizer $\mathbf{y}$, for each $(\xi_1, \xi_2) \in S$. This is the well known Legendre–Hadamard inequality (see Graves, 1939; Truesdell and Noll, 1965). Here $\mathbf{u}$ is an arbitrary three-dimensional vector and $\mathbf{v}$ is an arbitrary vector in the tangent plane of the reference surface Ω at the point (ξ_1, ξ_2). For membranes, the necessity of the Legendre–Hadamard condition for a general class of pressure potentials that includes (6, 7) has been demonstrated by Steigmann (1991).

For isotropic membranes, Pipkin (1986) has derived restrictions on the derivatives of the strain energy $W^*(\lambda, \mu)$ that, taken together, are equivalent to the Legendre–Hadamard inequality. These are:

$$W^*_\lambda \geqslant 0, \quad W^*_\mu \geqslant 0, \quad W^*_{\lambda\lambda} \geqslant 0, \quad W^*_{\mu\mu} \geqslant 0, \quad A \geqslant 0 \tag{11a}$$

and

$$(W^*_{\lambda\lambda} W^*_{\mu\mu})^{1/2} - W^*_{\lambda\mu} \geqslant B - A, \quad (W^*_{\lambda\lambda} W^*_{\mu\mu})^{1/2} + W^*_{\lambda\mu} \geqslant -B - A, \tag{11b}$$

where

$$A = (\lambda W^*_\lambda - \mu W^*_\mu)/(\lambda^2 - \mu^2) \quad \text{and} \quad B = (\mu W^*_\lambda - \lambda W^*_\mu)/(\lambda^2 - \mu^2). \tag{11c}$$

The first two of inequalities (11a) require that the principal stresses delivered by an energy minimizer be non-negative at every point in S. These inequalities have no counterparts in the Legendre–Hadamard condition for three-dimensional elasticity (Ogden, 1984). A particular strain energy function, adapted for use in membrane theory, will usually violate these restrictions in certain parts of the (λ, μ)-plane. Thus it is possible to formulate problems in the theory that have no energy minimizer.

To circumvent such difficulties while retaining the analytical simplicity of membrane theory, Pipkin (1986) introduced the notion of a relaxed energy density, defined so that the first two of inequalities (11a) are automatically satisfied for all $\lambda, \mu \geqslant 0$. The construction of this energy can be understood in terms of the behaviour of a strip under uniaxial tension. Thus suppose that a unit square of the membrane is stretched into a rectangle of dimensions $\lambda > 1$ and $\mu = g(\lambda)$, where $g(\lambda)$ is the unique solution of $W^*_\mu(\lambda, \cdot) = 0$. For a typical function $W^*(\lambda, \mu)$, a compressive force $W^*_\mu < 0$ would be required to make the strip narrower than $g(\lambda)$. The resulting deformation would then violate the second inequality in (11a).

Pipkin has shown that smooth deformations with $0 \leqslant \mu \leqslant g(\lambda)$ can be constructed as limits of sequences of finely wrinkled deformations containing closely spaced folds parallel to the tensile axis. The value of the strain energy at the limit of the sequence is equal to the original energy evaluated at $g(\lambda)$. Thus the relaxed energy is equal to:

$$\hat{W}(\lambda) = W^*(\lambda, g(\lambda)) \tag{12}$$

in that part of the (λ, μ)-plane where $\lambda > 1$ and $0 \leqslant \mu \leqslant g(\lambda)$. Since it is independent of μ in this region, it satisfies the second of (11a) as an equality. From the symmetry of the function $W^*(\lambda, \mu)$, it follows that the relaxed energy is $\hat{W}(\mu)$ if $\mu > 1$ and $0 \leqslant \lambda \leqslant g(\mu)$.

Deformations with stretches $0 \leqslant (\lambda, \mu) \leqslant 1$ can be constructed as limits of sequences with folds along both principal axes. The associated relaxed energy is $W^*(1, 1)$, which we take to be zero [Pipkin (1986)].

A typical strain energy function furnishes positive principal stresses in the region defined by $\lambda > g(\mu)$ and $\mu > g(\lambda)$, and is equal to its relaxation in this region if it satisfies the remaining inequalities in (11a, b). The inequalities also require that $f'(x) \geqslant 0$ and $f(x) \geqslant 0$ for $x > 1$, where $f = \hat{W}'(x)$ is the force-stretch relation in a wrinkled region. If these conditions are met, then the composite relaxed energy is:

$$W(\lambda, \mu) = \begin{cases} W^*(\lambda, \mu); & \lambda > g(\mu), \quad \mu > g(\lambda) \\ \hat{W}(\lambda); & \lambda > 1, \quad 0 \leqslant \mu \leqslant g(\lambda) \\ \hat{W}(\mu); & \mu > 1, \quad 0 \leqslant \lambda \leqslant g(\mu) \\ 0; & 0 \leqslant (\lambda, \mu) \leqslant 1. \end{cases} \tag{13}$$

The associated potential energy is given by (5), with W substituted in place of W^*.

In the present work we base our analysis on a strain energy function proposed by Varga for incompressible rubber-like materials (Varga, 1966). In the three-dimensional theory, the strain energy per unit of initial volume is:

$$U = 2G(\lambda_1 + \lambda_2 + \lambda_3 - 3); \quad \lambda_3 = (\lambda_1 \lambda_2)^{-1}, \tag{14}$$

where the λ_i are the principal stretches and G is the shear modulus for infinitesimal strains. The associated membrane energy may be approximated by setting $\lambda_1 = \lambda$, $\lambda_2 = \mu$ and taking λ_3 to be the stretch through the thickness. Thus:

$$W^*(\lambda, \mu) = 2G^*(\lambda + \mu + \lambda^{-1}\mu^{-1} - 3), \tag{15}$$

where $G^* = Gh$ and h is the initial thickness of the membrane. Then the transverse stretch in uniaxial tension is $\mu = g(\lambda)$, where:

$$g(\lambda) = \lambda^{-1/2}. \tag{16}$$

For any isotropic elastic material that is incompressible in bulk, the transverse stretch μ in simple tension is equal to the through-thickness stretch $1/\lambda\mu$. Thus (16) applies to every such material. For the Varga material, the strain energy in tension is then given by:

$$\hat{W}(\lambda) = W^*(\lambda, g(\lambda)) = 2G^*(\lambda + 2\lambda^{-1/2} - 3). \tag{17}$$

One may easily verify that the relaxed energy defined by eqn (13) and eqns (15)–(17) satisfies the Legendre–Hadamard inequality for all positive values of λ and μ, provided that $G^* > 0$. In that part of the (λ, μ)-plane where $W = W^*$, both principal stresses are positive and the membrane is said to be tense. In those parts where $W = \hat{W}$, one principal stress is tensile and the other vanishes identically. The associated stress state is called a *tension field*. The corresponding region of the membrane surface may be interpreted as finely wrinkled, though the theory furnishes only the average deformation in such a region. The distribution and spacing of wrinkles in an actual membrane are influenced by its bending stiffness, which is neglected in the present theory.

The stresses delivered by the relaxed energy vanish identically if both stretches are less than unity. The membrane may then be regarded as completely slack. Slack equilibrium states are not possible in the presence of lateral pressure.

4. KINEMATICS OF FLEXURE

Let the reference configuration of the membrane be a right circular cylinder of radius r, defined parametrically by:

$$\mathbf{x}(\theta, \xi) = r\mathbf{i}(\theta) + \xi\mathbf{k}, \tag{18}$$

where $\theta \in [0, 2\pi)$ is the azimuth, ξ is the axial length along a generator $\theta =$ constant, $\mathbf{k}$ is a fixed unit vector aligned with the generators and $\mathbf{i}(\theta)$ is the exterior unit normal to the cylindrical surface. We assume the membrane to be unstressed in this configuration.

We consider deformations that map points with coordinates $\xi_1 = \theta$, $\xi_2 = \xi$ onto the surface

$$\mathbf{y}(\theta, \xi) = \mathbf{r}(s(\xi)) + x(\theta)\mathbf{a}(s(\xi)) + y(\theta)\mathbf{b}(s(\xi)), \tag{19}$$

where $s(\xi)$ measures arc length along a particular base curve $\mathbf{r}(s)$ with principal normal $\mathbf{a}(s)$ and binormal $\mathbf{b}(s)$. This curve is defined by the requirement that the derivative:

$$s'(\xi) = \alpha, \tag{20}$$

where α is a positive constant to be specified in what follows. The principal curvature $\kappa(s)$ and principal normal $\mathbf{a}(s)$ of the base curve are defined by:

$$\kappa(s) = |\mathbf{t}'(s)|, \quad \mathbf{a}(s) = \kappa^{-1}\mathbf{t}'(s), \tag{21a, b}$$

where

$$\mathbf{t}(s) = \mathbf{r}'(s) \tag{21c}$$

is the unit tangent. The binormal is then given by:

$$\mathbf{b}(s) = \mathbf{t} \times \mathbf{a}. \tag{21d}$$

In its full generality, eqn (19) can be used to describe a particular class of deformations involving extension, flexure and twist of a tube in which every cross-section $\xi =$ constant suffers the same distortion. $x(\theta)$ and $y(\theta)$ are the rectangular coordinates of a point on the contour of the deformed cross-section. In the present work we consider the special case in which $\mathbf{b} =$ constant and $\kappa =$ constant. This corresponds to uniform flexure without twist in the plane containing $\mathbf{t}$ and $\mathbf{a}$, which we assume to be a plane of symmetry for the deformation. We also stipulate that the centre of curvature of the base curve coincide with the origin of the vector $\mathbf{r}(s)$ (Fig. 1).
Then:

$$\mathbf{a}(s) = -\kappa\mathbf{r}(s); \quad \kappa = \text{constant}, \tag{22}$$

and (19) becomes

$$\mathbf{y}(\theta, \xi) = [1 - \kappa x(\theta)]\mathbf{r}(s) + y(\theta)\mathbf{b}; \quad \mathbf{b} = \text{constant} \tag{23}$$

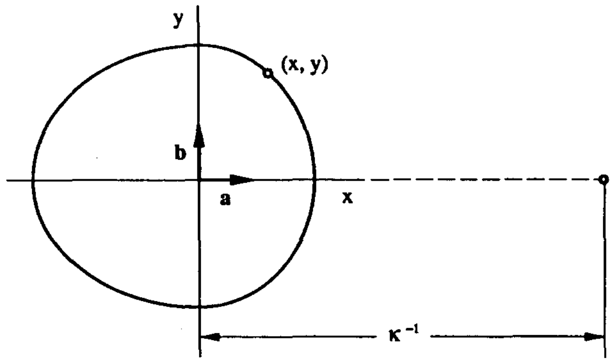

Fig. 1. Geometry of the deformed cross-section. The point with coordinates r, θ on the undeformed cross-section is displaced to the point with coordinates $x(\theta)$, $y(\theta)$ relative to a base curve with curvature κ.

To obtain the deformation gradient $\mathbf{F}$, we use eqns (20)–(23) to derive:

$$\mathrm{d}\mathbf{y}(\theta, \xi) = \alpha[1-\kappa x(\theta)]\mathbf{t}(s)\,\mathrm{d}\xi + [x'(\theta)\mathbf{a}(s) + y'(\theta)\mathbf{b}]\,\mathrm{d}\theta. \tag{24}$$

Now $\mathrm{d}\xi = \mathbf{k}\cdot\mathrm{d}\mathbf{x}$ and $\mathrm{d}\theta = r^{-1}\mathbf{j}(\theta)\cdot\mathrm{d}\mathbf{x}$, where $\mathbf{j}(\theta) = \mathbf{i}'(\theta) = \mathbf{k}\times\mathbf{i}(\theta)$ is the unit tangent to a parallel ξ = constant on the reference cylinder. Then comparison with (1) delivers $\mathbf{F}$ in the form (2) directly:

$$\mathbf{F} = \alpha(1-\kappa x)\mathbf{t}\otimes\mathbf{k} + [(x'/r)\mathbf{a} + (y'/r)\mathbf{b}]\otimes\mathbf{j}, \tag{25}$$

where r is the radius of the reference cylinder. To satisfy (3b) with $\mathbf{N} = \mathbf{i}(\theta)$, we have

$$\mathbf{L} = \mathbf{j}(\theta), \quad \mathbf{M} = \mathbf{k}. \tag{26}$$

Assuming that the centre of curvature of the base curve lies outside the deformed cross-section ($\kappa x < 1$), it then follows that

$$\lambda = [(x'/r)^2 + (y'/r)^2]^{1/2}, \quad \mu = \alpha(1-\kappa x) \tag{27a, b}$$

and

$$\mathbf{l} = \lambda^{-1}[(x'/r)\mathbf{a} + (y'/r)\mathbf{b}], \quad \mathbf{m} = \mathbf{t}. \tag{27c, d}$$

The first of eqns (27) gives the circumferential stretch of the cross-section and the second is the axial stretch of the longitudinal fibres of the tube. The associated principal vectors $\mathbf{l}$ and $\mathbf{m}$ are, respectively, tangential to the contour of the deformed cross-section and normal to the plane of the cross-section.

The parameter α may be interpreted as the axial stretch of the base curve $\mathbf{r}(s)$. Its specification amounts to an arbitrary choice of the location of the origin of the x-axis relative to the contour of the deformed cross-section. In particular, if the origin lies outside the contour, then there is no material line whose axial stretch is equal to α.

In the present work we select α to be the axial stretch associated with extension and expansion of the reference cylinder onto another cylinder of constant radius ρ. The deformed cylinder is defined by:

$$\mathbf{y}(\theta, \xi) = \rho\mathbf{i}(\theta) + z\mathbf{k}; \quad z = \alpha\xi, \tag{28}$$

apart from a uniform axial translation. This can be written in the form (19), with $\mathbf{r} = \alpha\xi\mathbf{k}$,

$\mathbf{a} = \mathbf{i}(0)$, $\mathbf{b} = \mathbf{j}(0)$ and

$$x(\theta) = \rho\cos\theta, \quad y(\theta) = \rho\sin\theta. \tag{29a, b}$$

The deformation gradient is easily obtained, and the circumferential and axial stretches are found to be:

$$\lambda = \rho/r \quad \text{and} \quad \mu = \alpha. \tag{30a, b}$$

Formally, these are the same as (27a, b) with $\kappa = 0$. However, (28) does not belong to the class (23) of flexural deformations as the centre of curvature is undefined if the curvature vanishes.

In the following section, α is determined from an elementary analysis of the equilibrium of the deformation (28) in the presence of a prescribed pressure. The analysis of the flexural response at this same pressure can then be based on (23) and attendant kinematical formulae. In effect, we regard the deformation as a finite flexure superposed on a primary equilibrium deformation that maps one cylinder onto another.

5. DETERMINATION OF THE PARAMETER α

Consider a cylindrical tube subjected to a net internal pressure P. We suppose that the ends of the tube are sealed in some manner to contain the pressure. Let the deformed surface be one of the family of cylinders described by (28). Then the strain energy is $2\pi r\ell W(\lambda, \mu)$, where λ and μ are given by (30a, b) and ℓ is the initial length of the cylinder. Here W is the relaxed energy defined by (13).

The load potential (6) is the sum of contributions from the lateral surface and the ends of the cylinder. On the lateral surface, the exterior unit normal is $\mathbf{n} = \mathbf{i}(\theta)$, and $\mathbf{y}\cdot\mathbf{n} = \rho$. Then from (7a, b) and (30a, b), $\chi = (1/3)\mu\lambda^2 r$, and the contribution to (6) is found to be $P(2\pi/3)\mu\lambda^2 r^2\ell$. We take the ends of the tube to be the cross-sectional planes on which $\xi = \pm\ell/2$. Then $\mathbf{n} = \pm\mathbf{k}$, $\mathbf{y}\cdot\mathbf{n} = \mu\ell/2$ and $\chi = (1/3)\mu J\ell/2$. The ends therefore contribute $2P(1/3)\mu\ell/2$ times the cross-sectional area. The total load potential is thus found to be $P\pi(r\lambda)^2\mu\ell$. As expected, the coefficient of P is the volume of a cylinder of length $\mu\ell$ and radius $\rho = \lambda r$.

We define the dimensionless strain energy w and the dimensionless pressure p by

$$w = W/2G^* \quad \text{and} \quad p = Pr/2G^*, \tag{31a, b}$$

where G^* is the material constant in (15). Then the potential energy (5) is proportional to

$$e(\lambda, \mu) = w(\lambda, \mu) - \tfrac{1}{2}p\mu\lambda^2. \tag{32}$$

The increment de induced by increments in λ and μ is:

$$de = (w_\lambda - p\lambda\mu)\,\mathrm{d}\lambda + (w_\mu - \tfrac{1}{2}p\lambda^2)\,\mathrm{d}\mu, \tag{33}$$

and the deformation is equilibrated if and only if this vanishes for arbitrary $\mathrm{d}\lambda$ and $\mathrm{d}\mu$, i.e.

$$w_\lambda = p\lambda\mu, \quad w_\mu = \tfrac{1}{2}p\lambda^2, \tag{34a, b}$$

where λ and μ are given by (30a, b). Thus the equilibrated cylinder is tense.

For Varga materials, the appropriate branch of the relaxed energy is

$$w(\lambda, \mu) = W^*/2G^* = \lambda + \mu + \lambda^{-1}\mu^{-1} - 3. \tag{35}$$

This can be used with (34a, b) to derive a pair of algebraic equations for λ and μ. Elimination of μ from this pair yields:

$$2p\lambda^5+p^2\lambda^4+4(1-\lambda^3+\lambda^2 p) = 0, \tag{36}$$

and either of eqns (34) then gives:

$$\mu = [2\lambda^{-1}(2-p\lambda^2)^{-1}]^{1/2}, \tag{37}$$

provided that $p\lambda^2 < 2$.

Regarding (36) as an equation for p, we find that there is one positive root:

$$p = [\lambda^{-4}(2+\lambda^3)^2+4\lambda^{-1}(1-\lambda^{-3})]^{1/2}-\lambda^{-2}(2+\lambda^3). \tag{38}$$

This furnishes the equilibrium value of the pressure associated with a particular value of the radius (Fig. 2).

Inspection of Fig. 2 shows that the equilibrium pressure has a maximum value. Furthermore, there are two equilibrium values of λ associated with each pressure between zero and the maximum pressure. Alternatively, it may be shown that for p between zero and the maximum value, (36) has three real roots λ; two of them are greater than unity and the third is negative. The latter root is therefore irrelevant. For each of the two equilibrium stretches λ, the corresponding stretch μ is given by (37).

A partial analysis of the stability of the two solutions can be based on (32): If an equilibrium state defined by (34) is stable, then it is necessary that the matrix of second derivatives of $e(\lambda, \mu)$ be positive semi-definite at the associated values of λ and μ, i.e.

$$e_{\lambda\lambda} \geqslant 0, \quad e_{\mu\mu} \geqslant 0, \quad e_{\lambda\lambda}e_{\mu\mu}-e_{\lambda\mu}^2 \geqslant 0. \tag{39a, b, c}$$

With p fixed, it follows from (32) and (35) that the second of inequalities (39) is automatically satisfied. The first and third of (39) can be reduced with the aid of (37) to:

$$p\lambda^2 \leqslant 1 \quad \text{and} \quad (3/2)\lambda^{-3}(2-p\lambda^2)-p^2\lambda^2 \geqslant 0, \tag{40a, b}$$

respectively. For the three values of p indicated in Fig. 2, we find that these inequalities are satisfied in the strict sense at points on the ascending branch of the function (38), and violated at points on the descending branch. The points on the latter branch are therefore unstable. However, our analysis has been restricted to deformations of the form (28), and we have not proved that points on the ascending branch correspond to stable configurations.

A comprehensive study of the bifurcation of pressurized cylindrical membranes has

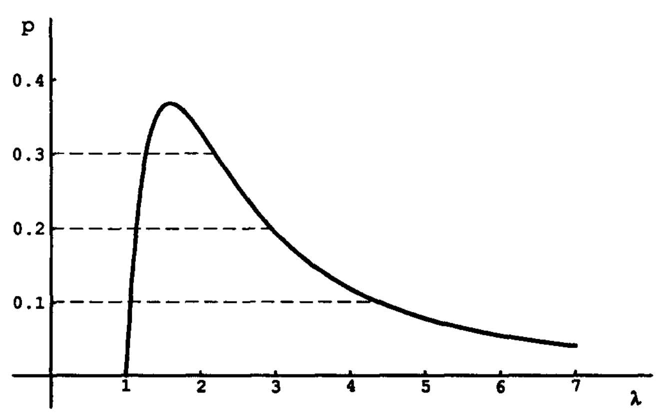

Fig. 2. Equilibrium pressure–radius response for a cylinder with sealed ends (Varga strain energy function).

been carried out by Haughton and Ogden (1979). They showed that for cylinders with sealed ends, inflated by a controlled pressure, bifurcation occurs at values of λ slightly greater than the value associated with the maximum pressure. They also proved that bifurcation occurs *at* the maximum pressure for cylinders of infinite length. These results suggest that points on the ascending part of the curve in Fig. 2 are locally stable in the sense that the second variation of the potential energy is positive definite for arbitrary nonrigid perturbations of the displacement. Stability is not assured, however, because the absence of bifurcation, while necessary for local stability, is not sufficient. Nevertheless the second variation is positive definite in the restricted class of deformations defined by (28), and we shall describe points on the ascending branch as locally stable in this sense.

More recently, Kyriakides and Chang (1991) demonstrated that for materials whose equilibrium $p-\lambda$ curves are qualitatively similar to Fig. 2, the postbifurcation response is dominated by a localized bulging mode reminiscent of an aneurism. However, their analysis presumes a prescription of volume rather than pressure.

In this work we consider three values of the prescribed pressure as indicated in Fig. 2. For each pressure the associated value of λ is obtained from the ascending branch of the curve and α is then identified with the value of μ computed from (37).

We remark that these configurations cannot be absolutely stable for Varga materials because the energy defined by (32) has no global minimum. For example, in deformations with $\mu = 1$ and $\lambda > 1$, the relaxed strain energy is given by (35) and the potential energy:

$$e = \lambda(1+\lambda^{-2}-\tfrac{1}{2}p\lambda)+\text{constant} \tag{41}$$

has no finite lower bound for $\lambda \in (1, \infty)$. This is due to the failure of Varga materials to satisfy suitable growth conditions for large stretches. But for the stretches furnished by the locally stable states, the behaviour of Varga materials is in close quantitative agreement with biaxial data on thin rubber sheets (Varga, 1966).

6. EQUILIBRIUM OF THE BENT TUBE

For deformations involving flexure, the load potential is obtained by combining (6, 7) with (23). From (3a, 7a, b) and (23, 27) we find that:

$$r\chi = (1/3)\alpha(1-\kappa x)[(x-\kappa^{-1})y'-yx'], \tag{42}$$

on the lateral surface of the tube. As in Section 5 we assume that the ends of the tube lie in the cross-sectional planes $\xi = \pm\ell/2$. Position $\mathbf{y}$ in these planes is given by (23), where x and y now range over the cross-sectional area and $s = \pm\alpha\ell/2$. The normals to these planes are $\mathbf{t}(\alpha\ell/2)$ and $-\mathbf{t}(-\alpha\ell/2)$, respectively, and these are orthogonal to $\mathbf{y}$. Thus the ends make no contribution to the load potential.

The relaxed energy $W(\lambda, \mu)$ may, by virtue of (27), be regarded as a function of $x(\theta)$, $x'(\theta)$ and $y'(\theta)$, depending parametrically on r, κ and α: $W = \bar{W}(x;\, x',\, y')$. Thus the potential energy per unit initial length of the cylinder is the functional of x and y defined by:

$$E/\ell = \int_0^{2\pi} F(x, y; x', y')\,\mathrm{d}\theta, \tag{43a}$$

where

$$F/r = \bar{W}-P\chi. \tag{43b}$$

Here we regard P as prescribed. The parameter α is then uniquely determined from the analysis of Section 5, and is therefore also prescribed. Then for a particular value of κ, the equilibrium equations for the bent tube are simply the Euler–Lagrange equations furnished by (43a):

$$F_x = (F_{x'})', \quad F_y = (F_{y'})'. \tag{44a, b}$$

In view of (27a, b) we have:

$$\left.\begin{aligned} \bar{W}_x &= -\alpha\kappa W_\mu, \quad \bar{W}_{x'} = r^{-1}\lambda^{-1}(x'/r)W_\lambda \\ \bar{W}_{y'} &= r^{-1}\lambda^{-1}(y'/r)W_\lambda \end{aligned}\right\}. \quad \text{and} \tag{45a, b}$$

From these and (42) it follows after some manipulation that the Euler-Lagrange equations (44a, b) reduce to:

$$\left.\begin{aligned} P\alpha(1-\kappa x)y' &= -(\lambda^{-1}W_\lambda x'/r)' - r\alpha\kappa W_\mu \\ P\alpha(1-\kappa x)x' &= (\lambda^{-1}W_\lambda y'/r)' \end{aligned}\right\}, \quad \text{and} \tag{46a, b}$$

respectively. These equations were obtained elsewhere (Steigmann, 1990b), without reference to variational principles, by specializing the general equations of membrane theory to deformations of the form (28).

Henceforth we shall base our formulation on the dimensionless curvature

$$k = \kappa r/2 \tag{47}$$

and the dimensionless coordinates

$$u(\theta) = x(\theta)/r, \quad v(\theta) = y(\theta)/r. \tag{48a, b}$$

In terms of these variables, eqns (27a, b) become:

$$\lambda = [(u')^2 + (v')^2]^{1/2} \quad \text{and} \quad \mu = \alpha(1-2ku), \tag{49a, b}$$

and eqns (46a, b) can be written:

$$\left.\begin{aligned} p\alpha(1-2ku)v' &= -(\lambda^{-1}w_\lambda u')' - 2\alpha k w_\mu \\ p\alpha(1-2ku)u' &= (\lambda^{-1}w_\lambda v')', \end{aligned}\right\} \tag{50a, b}$$

where w and p are the dimensionless energy and pressure defined in (31a, b).

The second of eqns (50) can be integrated once to obtain:

$$\lambda^{-1}w_\lambda v' - p\alpha u(1-ku) = a, \tag{51}$$

where a is a constant. Another integral of this system follows from the Euler–Lagrange equations (44) and the fact that the function F does not depend explicitly on the independent variable θ. Thus

$$F - x'F_{x'} - y'F_{y'} = \text{constant}. \tag{52}$$

According to (42), χ is a homogeneous, affine function of x' and y'. Thus the combination $\chi - x'\chi_{x'} - y'\chi_{y'}$ vanishes identically. Using (45a, b), we then find that (52) can be put into the form:

$$w - \lambda w_\lambda = b; \quad b = \text{constant}. \tag{53}$$

From the form of the function μ in (49b), one would expect that for k larger than some critical value depending on α, a tension field will develop on that part of the membrane

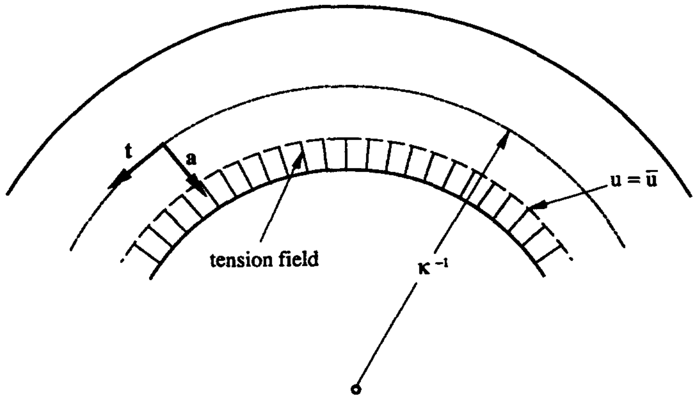

Fig. 3. Development of wrinkles on the deformed membrane.

for which $u \geqslant \bar{u}$, where $\bar{u}$ is the value of u at which the equality $\mu = g(\lambda) = \lambda^{-1/2}$ is satisfied. Accordingly, we seek a solution in which the membrane is either fully tense or partly tense and partly wrinkled. In the latter case, $\lambda > g(\mu)$ and $\mu > g(\lambda)$ in the region defined by $u < \bar{u}$, while $\lambda > 1$ and $\mu \leqslant g(\lambda)$ in the region $u \geqslant \bar{u}$. For Varga materials, the relevant branches of the dimensionless relaxed energy are:

$$w(\lambda, \mu) = \begin{cases} \lambda + \mu + \lambda^{-1}\mu^{-1} - 3, & u < \bar{u} \\ \lambda + 2\lambda^{-1/2} - 3, & u \geqslant \bar{u}. \end{cases} \tag{54}$$

The procedure for determining the parameters a, b and $\bar{u}$ is presented in Sections 7 and 8. If it is found that $\bar{u}$ exceeds the largest value of u in a particular deformation, then the membrane is judged to be tense and the first branch of (54) is used. Otherwise the membrane is partly wrinkled and both branches are operative in their respective domains of definition. The trajectories of the wrinkles are curves that lie in the cross-sectional planes ξ = constant (Fig. 3).

We have shown elsewhere (Steigmann, 1990a) that in equilibrium deformations the deformation gradient is continuous at the boundary between a tension field and a tense part of the membrane, unless the boundary happens to be a tension trajectory. In the present problem the boundaries are the generators $\theta = \pm\bar{\theta}$ on the reference surface corresponding to $u = \bar{u}$ in the deformed configuration. These are orthogonal to the tension trajectories. Thus the deformation gradient is continuous. From this result it follows that in the presence of wrinkling, the left hand sides of (51) and (53) are continuous at $u = \bar{u}$. The latter equations are therefore valid in both the tense and wrinkled parts of the membrane with the same values of the pair (a, b).

7. REDUCTION TO QUADRATURES

The integrals (51) and (53) of the equations of equilibrium may be used to reduce the analysis to quadratures. To proceed we first use (53, 54) to obtain $\lambda = \Lambda_t(u; b)$ in the tense part of the membrane, where:

$$\Lambda_t(u; b) = 2/[\mu(u)(b+3-\mu(u))], \tag{55a}$$

and $\mu(u)$ is the function defined by (49b). In the wrinkled part of the membrane we obtain $\lambda = \Lambda_w(b)$, where:

$$\Lambda_w(b) = (1+b/3)^{-2}. \tag{55b}$$

Thus λ is constant in the wrinkled region.

To locate the boundary $u = \bar{u}$ between the tense and wrinkled regions, we use the positive root of (55b) to calculate $g(\Lambda_w) = \Lambda_w^{-1/2}$. Setting this equal to μ at the boundary, we find from (49) that:

$$\bar{u}(b) = (1-\alpha^{-1}\Lambda_w^{-1/2})/2k. \tag{56}$$

Thus we may write:

$$\lambda = \Lambda(u;b) \equiv \begin{cases} \Lambda_t(u;b), & u \leqslant \bar{u}(b) \\ \Lambda_w(b), & u > \bar{u}(b). \end{cases} \tag{57}$$

It follows from the definition of the relaxed energy that the continuity condition $\Lambda_t(\bar{u};b) = \Lambda_w(b)$ is automatically satisfied. The procedure used to determine if the deformation is partly wrinkled or fully tense is given in Section 8. In the latter case $\Lambda = \Lambda_t$ throughout the membrane.

Next we use (57) to define:

$$H(u;b) \equiv \Lambda^{-1} w_\lambda[\Lambda(u;b), \mu(u)]. \tag{58}$$

This is strictly positive on both branches of the energy. Then (51) can be used to obtain:

$$v'(\theta) = K(u(\theta);a,b), \tag{59a}$$

where

$$K(u;a,b) \equiv [a + p\alpha u(1-ku)]/H(u;b). \tag{59b}$$

As indicated previously, we assume the deformation to be symmetric with respect to the u-axis. Thus it is sufficient to consider the upper half-contour $v \geqslant 0$. We further suppose that the points with coordinates $(u, v) = (u_o, 0)$ and $(u_\pi, 0)$ correspond to the generators $\theta = 0$ and π, respectively, on the reference cylinder (see Fig. 1). Thus:

$$u_o = u(0) \quad \text{and} \quad u_\pi = u(\pi); \quad u_o > u_\pi. \tag{60}$$

Assuming that $u'(\theta) \leqslant 0$ on the upper half-contour, we find from (49a), (57) and (59a) that:

$$u'(\theta) = -\{[\Lambda(u;b)]^2 - [K(u;a,b)]^2\}^{1/2}, \tag{61}$$

provided that $\Lambda^2 \geqslant K^2$. Then for fixed values of the parameters a, b, eqns (59a, 61) furnish a separable equation for the shape of the deformed half-contour:

$$\mathrm{d}v/\mathrm{d}u = -K(u;a,b)/\{[\Lambda(u;b)]^2 - [K(u;a,b)]^2\}^{1/2}. \tag{62}$$

From symmetry and continuity it follows that $\mathrm{d}u/\mathrm{d}v$ vanishes at $\theta = 0, \pi$. Thus:

$$[\Lambda(u;b)]^2 = [K(u;a,b)]^2 \quad \text{at } u = u_o, u_\pi. \tag{63}$$

Both solution branches of this equation are used to determine u_o and u_π in terms of the parameters a and b. In the next section we introduce selection criteria for u_o and u_π. Exhaustive numerical experiments indicate that the selection criteria always deliver a unique pair (u_o, u_π) of admissible roots. We do not know if this result is merely fortuitous, however. In this connection we remark that eqns (63) become very complicated if more complex strain energies are used. Thus our conclusions may not apply in general.

Once u_o and u_π have been determined, eqn (62) may be combined with the symmetry conditions $v = 0$ at u_o and u_π to obtain:

$$f_1(a,b) = 0, \tag{64a}$$

where

$$f_1(a,b) \equiv \int_{u_\pi(a,b)}^{u_o(a,b)} \frac{K(u;a,b)\,\mathrm{d}u}{\{[\Lambda(u;b)]^2-[K(u;a,b)]^2\}^{1/2}}. \tag{64b}$$

This places a restriction on the parameters a, b. To obtain a second restriction, we define $\hat{\theta}(u)$ to be the inverse of the function $u(\theta)$. Using (61), the requirement $\hat{\theta}(u_\pi)-\hat{\theta}(u_o) = \pi$ can then be put into the form:

$$f_2(a,b) = 0, \tag{65a}$$

where

$$f_2(a,b) \equiv \int_{u_\pi(a,b)}^{u_o(a,b)} \frac{\mathrm{d}u}{\{[\Lambda(u;b)]^2-[K(u;a,b)]^2\}^{1/2}} - \pi. \tag{65b}$$

If wrinkling is present, then the integrals are the sums of integrals over the intervals $[u_\pi, \bar{u})$ and $(\bar{u}, u_o]$, where $\bar{u}$ is given by (56). The procedure for obtaining a and b is based on numerical integration of (64b) and (65b) (Section 8).

The function $\hat{\theta}(\cdot)$ is determined by integrating (61):

$$\hat{\theta}(u) = \int_u^{u_o(a,b)} \frac{\mathrm{d}t}{\{[\Lambda(t;b)]^2-[K(t;a,b)]^2\}^{1/2}}, \quad u \in (u_\pi, u_o). \tag{66}$$

This is monotone and strictly decreasing on its domain. It therefore furnishes the function $u(\theta)$, $\theta \in (0, \pi)$, implicitly. The shape of the deformed half-contour is obtained by integrating (62):

$$v = \int_u^{u_o(a,b)} \frac{K(t;a,b)\,\mathrm{d}t}{\{[\Lambda(t;b)]^2-[K(t;a,b)]^2\}^{1/2}}, \quad u \in (u_\pi, u_o). \tag{67}$$

These results can be used to find the coordinates (u, v) corresponding to a generator $\theta = \text{constant}$.

To establish the existence of the integrals in (64b, 65b-67), we recall that u_o and u_π are roots of the equations $\Lambda = K$ and/or $\Lambda = -K$. Because these are polynomials in the variable u, it follows that for u near u_π we have:

$$\sqrt{\Lambda^2-K^2} \sim c\varepsilon^{1/2}, \quad \varepsilon \to 0^+, \tag{68}$$

where c is a nonzero constant and $\varepsilon = u - u_\pi$. A similar result holds for u near u_o, with $\varepsilon = u_o - u$. Thus the integrands have integrable singularities at the limits u_o and u_π. Moreover, the selection criteria for u_o and u_π always deliver successive roots of (63). This implies that the integrands are finite in the open interval (u_π, u_o).

8. NUMERICAL SOLUTION

(a) *Selection criteria for u_o and u_π.* The integration limits u_o and u_π are determined by solving eqns (63) for fixed values of the data (p, k) and the parameters (a, b). In a tense part of the membrane, eqns (55a, 58, 59b) and (54) can be used to reduce (63) to a pair of cubic equations:

$$a+p\alpha u(1-ku) = \pm[1-\mu(b+3-\mu)^2/4], \tag{69a}$$

where $\mu(u)$ is given by (49b). In a wrinkled region we obtain the quadratic equations:

$$a+p\alpha u(1-ku) = \pm(1-\Lambda_w^{-3/2}), \tag{69b}$$

where Λ_w is defined in (55b).

Apart from the requirement that the numbers u_o and u_π be real, certain additional restrictions can be stated immediately, namely:

$$u_o > u_\pi, \quad 2ku_o \leqslant 1 \quad \text{and} \quad \alpha(1-2ku_\pi) > g[\Lambda_t(u_\pi; b)], \tag{70}$$

where $g(\cdot)$ is defined by (16). The second inequality is needed to ensure that the minimum longitudinal stretch μ is non-negative. The third must be satisfied if $w_\mu > 0$ at $u = u_\pi$. This in turn follows from our hypothesis about the structure of solutions (Section 6). If violated, the membrane would be unable to equilibrate the force due to the pressure acting on its cross-section. Thus u_π belongs to the set of roots of (69a). Our hypothesis also implies that $w_\lambda > 0$ throughout the membrane. However, there is no *a priori* reason to suppose that the stretches defined by (49b) and (57) satisfy this requirement. Thus we impose the additional restrictions:

$$w_\lambda[\Lambda(u; b), \mu(u)] > 0 \quad \text{at } u_o, u_\pi, \tag{71}$$

where w_λ is computed from (54) and Λ is obtained from (57).

We find that the selection criteria (70, 71) yield a unique pair of roots u_o and u_π. We also find that $\Lambda^2 > K^2$ for $u \in (u_\pi, u_o)$. Thus the assumptions underlying the development in Section 7 are verified in the course of solution.

(b) *Determination of the parameters a, b.* To complete our analysis we require solutions (a, b) of the system (64a, 65a). We find through numerical experimentation that for each value of the parameter b belonging to a certain interval, there is just one value of the parameter a that solves (64a). Thus a function $a^*(\cdot)$ can be constructed at discrete points such that $a = a^*(b)$ in this interval. Then the problem reduces to finding b such that:

$$f^*(b) \equiv f_2(a^*(b), b) = 0, \tag{72}$$

where f_2 is defined in (65b).

The procedure requires some estimate of the intervals in which solutions (a, b) are to be sought. For example, if the curvature k is very small, then we expect to find solutions very nearly equal to those associated with the purely cylindrical deformations described by (29). This of course presumes that (29) furnishes solutions of eqns (51) and (53) with $k = 0$. To demonstrate this, we use (48) to write (29a, b) as:

$$u(\theta) = \lambda\cos\theta, \quad v(\theta) = \lambda\sin\theta, \tag{73a, b}$$

where λ is given by (30a). Then the left hand side of (51) becomes:

$$\lambda^{-1}w_\lambda v' - p\alpha u = (w_\lambda - p\lambda\alpha)\cos\theta, \tag{74}$$

which vanishes by virtue of (30b) and (34a). Thus (51) is satisfied with $a = 0$. Since the stretches are constants in cylindrical deformations, we have $w - \lambda w_\lambda = \text{const.}$, and (53) furnishes the appropriate value of the parameter b.

To illustrate the solution procedure, we consider the example $p = 0.3$ in some detail. For purely cylindrical deformations, the analysis described in Section 4 furnishes the solutions:

$$(\lambda, \mu)_1 = (1.26, 1.02) \quad \text{and} \quad (\lambda, \mu)_2 = (2.19, 1.28). \tag{75a, b}$$

These correspond to the locally stable and unstable configurations, respectively. Thus we set $\alpha = \mu_1 = 1.02$. The associated value of the parameter b is $b_1 = -0.43$. Thus for small curvatures we expect to find solutions (a, b) near the point $(0, -0.43)$, emanating from the stable solution (75a). Moreover, in view of (55b) the parameter b must be negative if the curvature is sufficient to cause partial wrinkling with $\lambda > 1$ in the wrinkled region. Accordingly, we consider an interval of negative values of b containing b_1.

Next we prescribe a small positive value of the curvature k. Assuming this curvature to be insufficient to induce wrinkling, we then obtain the real roots of (69a) for fixed b and for values of the parameter a belonging to an interval containing zero. For each value of a, the value of u_o delivered by the selection criteria is used to check that $u_o < \bar{u}\,(b)$, where $\bar{u}$ is obtained from (56). If this condition is violated then the membrane is assumed to be partly wrinkled and u_o is obtained from the set of real roots of (69b), in accordance with the selection criteria and the additional requirement $u_o \geqslant \bar{u}(b)$. This procedure generates the sets $\{u_o(a_i, b), u_\pi(a_i, b)\}$; $i = 1, \ldots, n$, where n is the number of values of the parameter a considered for a particular value of b. For each set, the numbers $f_1(a_i, b)$ may be computed from (64b) by numerical integration. We find one value of a that satisfies (64a) within a specified tolerance for each fixed b. For $k = 0.08$ the function $a = a^*(b)$ delivered by this procedure is shown in Fig. 4. The function $f^*(b)$ may then be computed from (65b). This is also plotted in Fig. 4.

We find two solutions of (72) in the interval considered. The first solution pair (a, b) is quite close to the solution furnished by the locally stable cylindrical deformation. For the second solution, the parameter b is nearer to the value $b_2 = -1.01$ obtained by substituting the unstable cylindrical deformation (75b) into (53). The parameter a is not close to zero, however. This is to be expected since the value of α used in (51) is obtained from the stable cylinder, and the stretches (λ_2, α) do not satisfy (34a). Because $\alpha < \mu_2$, the effect is simply to displace the cross-section of the second solution to the left relative to the origin of the (u, v)-axes. Thus for fixed p, α and k, the curvature of a generator $\theta =$ constant is larger in the first solution than in the second.

To compute the response of the membrane as a function of curvature at fixed pressure, we prescribe small additional increments in k and repeat the foregoing solution procedure at each step. We always obtain two solution pairs (a, b), each of which appears to depend

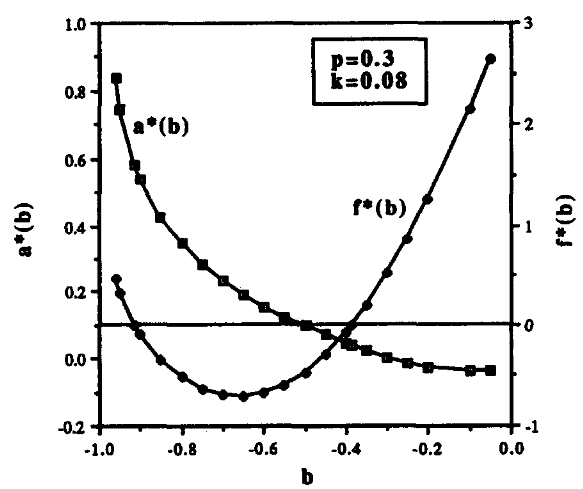

Fig. 4. Determination of the parameters (a, b) at $(p, k) = (0.3, 0.08)$. The function $a^*(b)$ furnishes the values of the parameter a that yield $v = 0$ at $u = u_o, u_\pi$. The requirement that these points correspond to $\theta = 0$ and π, respectively, is expressed by the condition $f^*(b) = 0$.

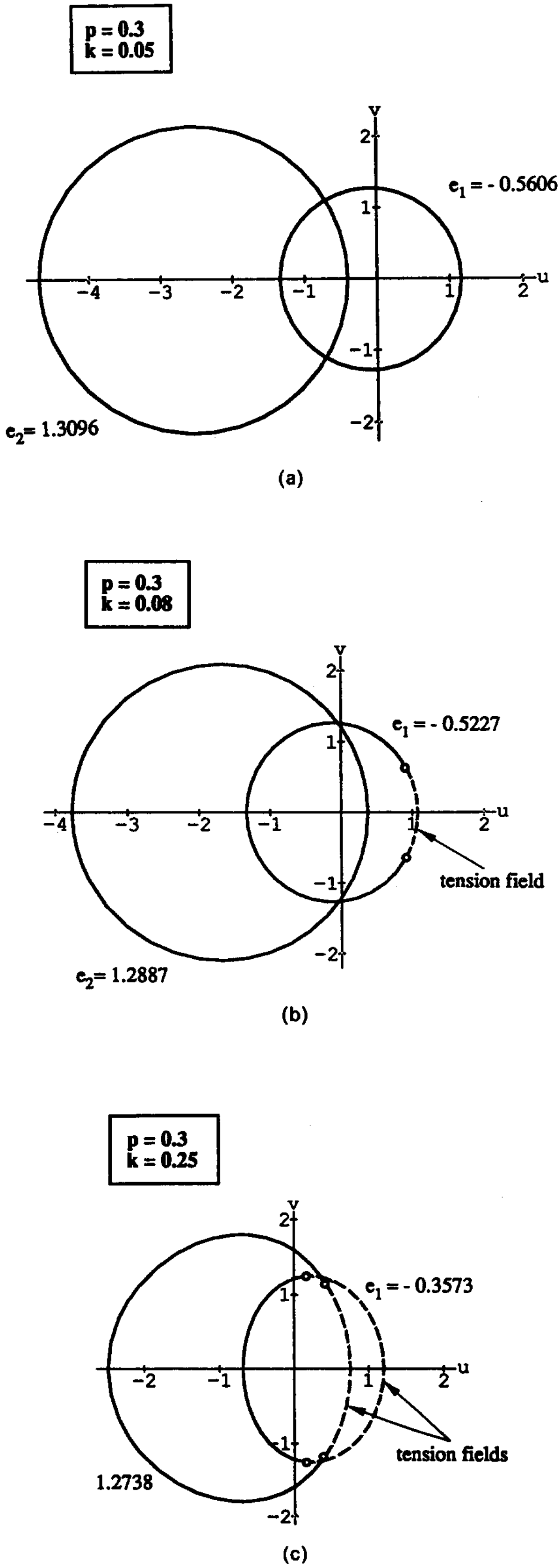

Fig. 5. Cross-sections of the two equilibrium configurations at $p = 0.3$ and three values of the dimensionless curvature, k, of the base curve. The shift of the larger, higher-energy cross-section is due to the definition of the parameter α that fixes the location of the base curve. In (a) both solutions are completely tense. The progressive development of wrinkling is shown in (b) and (c).

continuously on the curvature k. Furthermore, the solutions closely approximate those furnished by the two cylindrical deformations for small values of k. The first solution reduces to the locally stable cylinder as k approaches zero, and is always associated with the smaller absolute value of b. The larger value corresponds to the unstable cylinder. Thus we expect that there is no curvature for which the second solution is even locally stable.

Further evidence in support of this conjecture is furnished by a comparison of the potential energies of the two solutions. Using (31), (47) and (48), the potential energy (43a) may be shown to be proportional to:

$$e = \int_{u_\pi}^{u_o} \{w[\Lambda(u;b), \mu(u)] - p\alpha u(1-ku)K(u;a,b)\} \frac{du}{\sqrt{\Lambda^2 - K^2}}, \tag{76}$$

(see Appendix A). For equal values of k, we find that the first solution always has the smaller energy, despite the larger curvatures of the generators. However, it may be more meaningful to compare energies at equal values of the curvatures of the centroidal axes, for example. In particular, it may be difficult or impossible to control the curvature of the base curve in practice. Thus we regard the comparison as merely indicative.

Equation (67) was used to generate the shapes of the deformed cross-sections corresponding to $p = 0.3$ and $k = 0.05$, 0.08 and 0.25. These are shown in Figs 5(a), 5(b) and 5(c), respectively, where the energies of the two solutions are also indicated.

For $k = 0.08$, the stress distributions w_λ and w_μ furnished by the low-energy solution are shown in Fig. 6 as functions of the coordinate u. The first stress is the hoop tension transmitted across the generators of the tube. The second is the tension along the generators. It vanishes identically in a wrinkled region. We find that once wrinkling has initiated at $u = u_o$, it continues to cover an increasing portion of the membrane as k increases. The internal pressure prevents complete wrinkling, however.

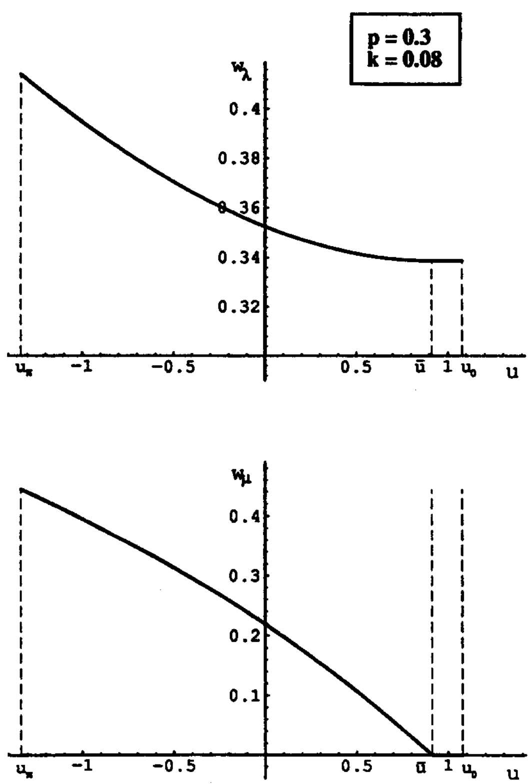

Fig. 6. Distributions of hoop tension w_λ and axial tension w_μ furnished by the low-energy solution at $(p, k) = (0.3, 0.08)$. The membrane is tense in the interval $[u_\pi, \bar{u})$ and wrinkled in the interval $(\bar{u}, u_o]$.

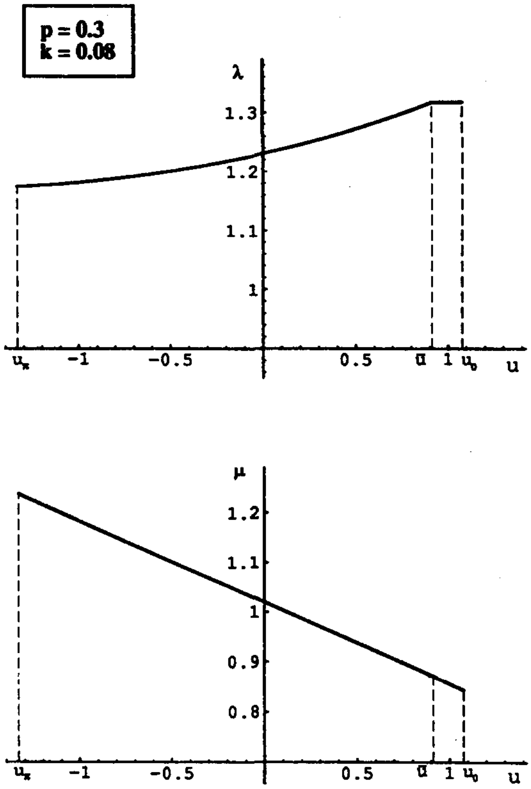

Fig. 7. Distributions of hoop stretch λ and axial stretch μ furnished by the low-energy solution at $(p, k) = (0.3, 0.08)$. The hoop stretch is constant in the interval $(\bar{u}, u_o]$ corresponding to the wrinkled region.

Finally, we remark that the stretches λ and μ associated with the low-energy solution fall within the range in which the Varga material is known to provide good quantitative agreement with uniaxial and biaxial data on rubber (Varga, 1966). The stretch distributions corresponding to $k = 0.08$ are shown in Fig. 7. The second solution furnishes principal stretches that are well outside this range, however.

9. FORCE AND MOMENT FORMULAE

In this final section we use solutions (a, b, u_o, u_π) of the foregoing equilibrium problem to compute the resultant force and moment due to the stress distributed over a cross-section ξ = constant.

(a) *Resultant force*. The stress-resultant **T** in the membrane is given by (4), with the relaxed energy W substituted in place of W^*. Then from (26) and (27c, d), the force per unit initial arc length acting on a cross-section is $\mathbf{Tk} = W_\mu \mathbf{t}$, and the resultant force is $F\mathbf{t}$, where:

$$F = r \int_0^{2\pi} W_\mu \, d\theta. \tag{77}$$

For equilibrium deformations, eqn (46a) leads to $F = PA^*$, where:

$$A^* = \oint x \, dy = r^2 \oint u \, dv, \tag{78}$$

and the integrals are evaluated on the contour of the deformed cross-section. It follows from Green's theorem that A^* is simply the cross-sectional area of the deformed membrane. Then, as expected, F is the resultant force due to the pressure acting on the cross-section. To compute it we use eqn (A.6) of Appendix A to obtain:

$$A^* = 2r^2 I_1\,; \quad I_1 = \int_{u_\pi}^{u_o} \frac{uK\,\mathrm{d}u}{\sqrt{\Lambda^2 - K^2}}. \tag{79}$$

(b) *Resultant bending moment about the centroid.* Perhaps the most descriptive measure of the overall flexural response of the tube is the resultant moment associated with a given curvature at equilibrium. Because of the force F, the value of the moment will, of course, depend on the origin used for its measurement. A convenient choice is the origin of the (u, v)-axes. This corresponds to the base curve whose curvature k is prescribed. However, the selection of the base curve itself is only a matter of convenience, having no intrinsic significance insofar as the deformation of the membrane is concerned. Our particular choice was motivated by the need for *a priori* estimates of the parameters a and b in eqns (51, 53), for states differing only slightly from purely cylindrical deformations.

Instead, we compute the moment relative to the centroid of the deformed cross-section. This furnishes a description of the response that is intrinsic to the problem, i.e. independent of the choice of base curve. Because the resultant force is known, the moment relative to any other axis may then be determined.

The position $\mathbf{y}_c$ of a point on the centroidal axis is defined by:

$$A^*\mathbf{y}_c = \iint \mathbf{y}\,\mathrm{d}x\,\mathrm{d}y, \tag{80}$$

where $\mathbf{y}$ is given by (23) and the integral extends over the deformed cross-section. Then the moment relative to the centroid is:

$$\mathbf{M}_c = \mathbf{M}_o - PA^*\mathbf{y}_c \times \mathbf{t}, \tag{81a}$$

where

$$\mathbf{M}_o = r\int_0^{2\pi} W_\mu \mathbf{y} \times \mathbf{t}\,\mathrm{d}\theta, \tag{81b}$$

is the moment about the centre of curvature.

According to (22) and (23),

$$\mathbf{y}_c = (x_c - \kappa^{-1})\mathbf{a}(s), \tag{82}$$

where x_c is the x-coordinate of the centroid relative to the base curve. The y-coordinate vanishes due to the symmetry of the deformation. It follows as a consequence of Green's theorem and (48) that

$$A^*x_c = \frac{1}{2}\oint x^2\,\mathrm{d}y = \frac{1}{2}r^3\oint u^2\,\mathrm{d}v. \tag{83}$$

Then from (A.6) of Appendix A we obtain:

$$A^*x_c = r^3 I_2\,; \quad I_2 = \int_{u_\pi}^{u_o} \frac{u^2 K\,\mathrm{d}u}{\sqrt{\Lambda^2 - K^2}}. \tag{84}$$

The curvature of the centroidal axis is $\kappa_c = |\mathbf{y}_c|^{-1} = (\kappa^{-1} - x_c)^{-1}$. We define the dimensionless curvature as in (47):

$$k_c = \kappa_c r/2. \tag{85a}$$

Then

$$k_c = (k^{-1} - 2x_c/r)^{-1}; \quad 2x_c/r = I_2/I_1. \tag{85b}$$

In Appendix B we use the equilibrium equations and the symmetry of the deformation to demonstrate that $\mathbf{M}_c = M_c\mathbf{b}$, where $\mathbf{b}$ is the normal to the plane of flexure and

$$M_c = 2G^* r^2 m_c. \tag{86a}$$

Here

$$m_c = p(k^{-1}I_1 - I_2) + \alpha^{-1}k^{-1}I_3 \tag{86b}$$

is the dimensionless moment, where

$$I_3 = \int_{u_\pi}^{u_0} H\sqrt{\Lambda^2 - K^2}\,\mathrm{d}u \tag{86c}$$

and H is the function of u defined in (58).

Equations (85b, 86b, c) were used to compute k_c and m_c as functions of k for the dimensionless pressures $p = 0.1, 0.2$ and 0.3. The calculated curves of m_c vs k_c are displayed in Fig. 8 for the solutions with the smaller potential energies. These results illustrate the stiffening effect of pressure on the flexural response. The low-energy solutions shown in Figures 5(a, b, c) correspond to $k_c = 0.049$, 0.078 and 0.286, respectively.

Certain qualitative features of the response are shared by solutions of shell theories that incorporate the effects of the bending stiffness of the material. The most striking of these is the occurrence of a maximum moment, preceded and followed by monotone behaviour (see Reissner and Weinitschke, 1963). The arrows indicate the onset of partial

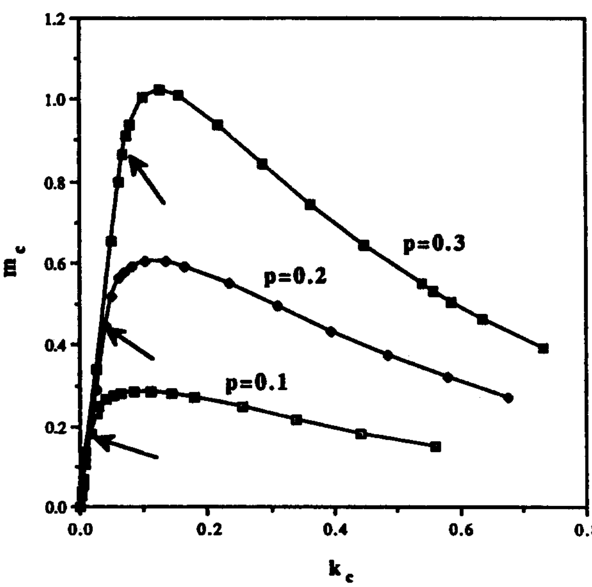

Fig. 8. Bending moment about the centroid as a function of the curvature of the centroidal axis at three values of the inflation pressure. Arrows indicate incipient wrinkling at $u = u_o$.

wrinkling, which is seen to occur before the maximum is attained. This is the condition corresponding to incipient wrinkling at $u = u_o$. Thereafter, wrinkling develops progressively with increasing curvature.

The moment–curvature response is analogous to the pressure–radius response of Fig. 2, which is known to be associated with localized radial bulging following bifurcation from a purely cylindrical deformation at or near the maximum pressure (Kyriakides and Chang, 1991). Thus we conjecture that, for a given pressure, our solution becomes unstable in any experimentally feasible loading program at some value of the curvature near the value corresponding to the maximum moment. The analogy with bulging suggests that the instability is characterized by a localization of curvature and an attendant variation of the cross-sectional shape along the axis of the tube. This supposition is supported by experimental data on the flexure of actual tubes (Emmerling, 1984). Such deformations do not belong to the class (19), however. Thus the present theory cannot be used to describe them.

The localized bending observed in thin-walled polyethylene tubes led Lukasiewicz and Glockner (1985) and Lukasiewicz and Balas (1990) to formulate an approximate *pneumatic-hinge* model of the collapse of pressurized cylinders. Our analysis would seem to provide further impetus for such theories. In contrast, the Stein–Hedgepeth theory for small bending deflections of partly wrinkled membranes predicts a stable moment–curvature relation that asymptotically approaches a finite upper bound (Stein and Hedgepeth, 1961). Koga (1972) developed a similar theory for small bending deformations superposed on a finite cylindrical deformation of a pressurized membrane. The latter theory accounts for the stiffening effect of pressure but exhibits a flexural response that is otherwise similar to that predicted by the Stein–Hedgepeth theory.

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APPENDIX A

According to (42, 43), the total energy is given by:

$$E/r\ell = \int_0^{2\pi} W\,d\theta - I, \tag{A.1}$$

where

$$I = \frac{P\alpha}{3r}\int_0^{2\pi}(1-\kappa x)\,[(x-\kappa^{-1})y' - yx']\,d\theta. \tag{A.2}$$

From (61) and the symmetry of the deformation with respect to the u-axis, it follows that:

$$d\theta = \mp du/\sqrt{\Lambda^2 - K^2} \tag{A.3}$$

on the upper and lower half-contours, respectively. Then the first term in (A.1) may be written:

$$\int_0^{2\pi} W\,d\theta = 4G^*\int_{u_\pi}^{u_0}\frac{w\,du}{\sqrt{\Lambda^2-K^2}}, \tag{A.4}$$

where w is the dimensionless strain energy.

The integrand in (A.2) may be reduced to $3x(1-\frac{1}{2}\kappa x)y'$, modulo an exact differential. Thus, in terms of the dimensionless coordinates u, v and the dimensionless curvature k, (A.2) becomes:

$$I = P\alpha r\oint u(1-ku)\,dv. \tag{A.5}$$

From (62) and the symmetry of the deformation, we have:

$$dv = \mp K\,du/\sqrt{\Lambda^2-K^2} \tag{A.6}$$

on the upper and lower half-contours. Then:

$$I = 4G^*p\alpha\int_{u_\pi}^{u_0} u(1-ku)\frac{K\,du}{\sqrt{\Lambda^2-K^2}}, \tag{A.7}$$

where p is the dimensionless pressure. On combining (A.4) and (A.7) with (A.1), we obtain $E/4G^*r\ell = e$, where e is given by eqn (76).

APPENDIX B

On combining (81, 82) and (22, 23), we obtain:

$$\mathbf{M}_c = \mathbf{M}_o + PA^*(x_c - \kappa^{-1})\mathbf{b}, \tag{B.1}$$

where

$$\mathbf{M}_o = \left[r\int_0^{2\pi}(\kappa^{-1}-x)W_\mu\,d\theta\right]\mathbf{b} + \left(r\int_0^{2\pi} yW_\mu\,d\theta\right)\mathbf{a}. \tag{B.2}$$

For equilibrium deformations it follows from (46a) that:

$$r\alpha\kappa yW_\mu = -\frac{1}{2}P\alpha(1-\kappa x)(y^2)' - y(\lambda^{-1}W_\lambda x'/r)'. \tag{B.3}$$

Integrating the right hand side by parts, we find:

$$r\alpha\kappa\int_0^{2\pi} yW_\mu\,\mathrm{d}\theta = -P\alpha\kappa\left(\frac{1}{2}\oint y^2\,\mathrm{d}x\right)+2G^*r\int_0^{2\pi}\lambda^{-1}w_\lambda u'v'\,\mathrm{d}\theta, \tag{B.4}$$

where u, v are the dimensionless coordinates and w is the dimensionless strain energy. According to Green's theorem, the first term on the right is proportional to the y-coordinate of the centroid of the deformed cross-section. This vanishes by symmetry. Furthermore, it follows from (51) that the integrand in the second term is expressible as an exact differential. Thus the coefficient of $\mathbf{a}$ in (B.2) vanishes and $\mathbf{M}_o = M_o\mathbf{b}$, where:

$$M_o = \kappa^{-1}PA^* - r\int_0^{2\pi} xW_\mu\,\mathrm{d}\theta. \tag{B.5}$$

Then (B.1) reduces to $\mathbf{M}_c = M_c\mathbf{b}$, where:

$$M_c = PA^*x_c - r\int_0^{2\pi} xW_\mu\,\mathrm{d}\theta. \tag{B.6}$$

Finally, we use (46a) in the second term to eventually obtain:

$$M_c = PA^*(\kappa^{-1}-x_c)+2G^*r\alpha^{-1}\kappa^{-1}\int_0^{2\pi}\lambda^{-1}w_\lambda(u')^2\,\mathrm{d}\theta, \tag{B.7}$$

where A^* and x_c are defined by (78) and (83), respectively.

From (61) and (A.3) of Appendix A we have:

$$(u')^2\,\mathrm{d}\theta = \mp\sqrt{\Lambda^2-K^2}\,\mathrm{d}u \tag{B.8}$$

on the upper and lower half-contours. Then:

$$\int_0^{2\pi}\lambda^{-1}w_\lambda(u')^2\,\mathrm{d}\theta = 2\mathrm{I}_3, \tag{B.9}$$

where I_3 is defined in (86c). Next we use (79) and (84) to write:

$$PA^*(\kappa^{-1}-x_c) = 2G^*r^2p(k^{-1}I_1-I_2), \tag{B.10}$$

where p is the dimensionless pressure. Substitution of (B.9) and (B.10) into (B.7) then yields $M_c = 2G^*r^2m_c$, where m_c is given by (86b).

Equations (81a, b) are well-defined for the cylindrical deformations described by (28). With $\mathbf{t} = \mathbf{k}$, $\mathbf{y}_c = z\mathbf{k}$ and the equilibrium equation (34b), we obtain:

$$\mathbf{M}_c = rW_\mu\int_0^{2\pi}\mathbf{y}\times\mathbf{k}\,\mathrm{d}\theta. \tag{B.11}$$

This vanishes by virtue of (28) and (29). Equations (77), (30a) and (34b) may also be used to obtain $F = PA^*$, where $A^* = \pi\rho^2$.